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# ON HINDERED SETTLING OF PARTICLES OF DIFFERENT SIZES

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**Abstract**—The flow of a non-dilute fluid suspension is considered in which the dispersed phase consists of particles or droplets of different sizes. A phenomenological two-phase flow theory is formulated for both continuous and discrete distributions of particle sizes and illustrated by considering the batch settling of such a mixture. The volume fractions and particle distribution functions are determined, as well as the composition of the sedimentary layer.

## 1. INTRODUCTION

We consider here the motion of a fluid suspension in which the dispersed phase consists of droplets or particles of different sizes. The volume fraction of this phase is assumed sufficiently large for the hindering effects of the droplets on each other to be important. (Droplets and particles are used interchangeably for the time being.) The objective is a theory to determine the motion *and* the distribution of particle sizes.

Two-phase flows are intrinsic to many technological operations—sedimentation, fluidization, boiling, material processing, separation, and combustion, to name a few. Knowledge about these highly complex processes has come mainly from observation and experimentation, but in recent years theoreticians, aided by large computers, have developed elaborate models of such phenomena. Although theory is to an extent derived from correct conservation laws by various averaging procedures (Ishii 1975, Delhaye & Achard 1976), empirical correlations and "reasonable" hypotheses must be employed for the physics of non-dilute suspensions to be described properly. At the very least, relationships are always required for the stress and rate of strain in each phase and the momentum and the drag interactions between phases. The final system of equations, in number and structure, is formidable. Application, of necessity, is often made at once to the practical problems at hand using extensive computer programs (Harlow & Amsden 1975, Rivard & Torrey 1977), although the basic theoretical framework is not well understood. There is obviously need to evaluate theory in almost all respects and the solution of simple, "idealized" problems which can be subjected to experimental verification serves this purpose best. The settling of particles and droplets in a force field is a problem of this type which for these reasons has been widely studied. The present work which adds to this body of research deals first with a suspension of spherical particles of $n$ different sizes (but having the same material density). The focus here is on a reasonable drag law and its implications; the effects of apparent mass, acceleration, shear lift etc. (Zuber 1964) will be examined in detail later. The equations for a continuous distribution are obtained by passing to the limit as the number $n$ of different particle sizes becomes infinite. Finally, both formulations are used to solve anew the problem of the gravitational settling of a suspension (Kynch 1962, Smith 1965, 1966, Lockett & Al-Habbooby 1974).

## 2. FORMULATION FOR A DISCRETE SIZE DISTRIBUTION

A droplet (or particle) in our non-dilute suspension is assumed to be a small sphere whose diameter is one of only $n$ possibilities. Let $\lambda_s$, $\lambda_l$ be the smallest and largest droplet radii, then the sizes are ordered as follows:

$$\lambda_s = \lambda_n < \lambda_{n-1} < \cdots < \lambda_1 = \lambda_l.$$

If a continuous size distribution were divided into $n$ intervals, $\lambda_k$ would be identified as the average radius for particles in the $k$th diameter range.

The size of an irregularly shaped particle may be chosen as the radius of an equivalent sphere which in the same circumstances moves in the same way. Although such a measure might depend on the orientation and position of the droplet, size is considered fixed in this analysis. In addition, there is to be no mass exchange between phases, no coalescence of droplets and all fluids are incompressible.

No attempt is made to develop theory from the fundamental standpoint of statistical mechanics or by a systematic extension from dilute conditions, but the distribution or density of droplets as a function of position and time is of prime interest. For a finite number of distinct sizes, the distribution is described by the partial volume fractions $\alpha_k(\mathbf{r}, t)$, of spherical droplets of radius $\lambda_k$. A continuous range of particle diameters necessitates a volume distribution $\phi(\mathbf{r}, t;\lambda)$, to be defined in the next section. In either case, the effects of collisions and interactions of the droplets and particles are described solely in gross phenomenological terms, as for example a drag force that is dependent on volume fractions, position and time.

Identical droplets of the same radius can be viewed as a separate fluid phase. In this light, the suspension having $n$ different particle sizes consists of $n$ incompressible, dispersed phases which are distinguished by particle diameter only.

The equations of motion of such a multiphase suspension of incompressible fluids may be obtained by local instant time-averaging procedures, Delhaye & Achard (1976), Ishii (1975). The generalization from two to an $n$-phase mixture is implicit in these references. For incompressible fluids and under the conditions assumed, a complete description of the flow is obtained from the mass and momentum conservation laws:

$$\frac{\partial}{\partial t}\alpha_k\rho_k + \nabla \cdot \alpha_k\rho_k\mathbf{v}_k = 0, \quad k = 0, \ldots, n \qquad [2.1]$$

$$\frac{\partial}{\partial t}\alpha_k\rho_k\mathbf{v}_k + \nabla \cdot \alpha_k\rho_k\mathbf{v}_k\mathbf{v}_k = -\alpha_k\nabla p_k + \nabla \cdot \alpha_k\underset{\sim}{\pi}_k + \alpha_k\rho_k\mathbf{g} + \mathbf{M}_k. \qquad [2.2]$$

The continuous fluid phase is designated by index $k = 0$ when this is appropriate, but more commonly by subscript $c$ as for example, $p_c = p_0$.

The phase averaged variables, defined precisely in Ishii (1975), are velocity, $\mathbf{v}_k$; volume fraction, $\alpha_k$; stress, $\underset{\sim}{\pi}_k$; partial pressure, $p_k$; interfacial drag force, $\mathbf{M}_k$; body force, $\mathbf{g}$. In particular, $\rho_0 = \rho_c$ and $\rho_k = \rho_D$, the density of the material of the dispersed phase for $k = 1, \ldots, n$.

Formulas for $\underset{\sim}{\pi}_k$, $\mathbf{M}_k$ and $p_k$ must reflect the main effects of droplet interactions or particle collisions. We adopt a phenomenological viewpoint and use hypotheses which seem reasonable and consistent with the limited evidence available.

The pressure inside a droplet is assumed to be related to that just outside by the capillary law

$$p_k = p_c + 2\sigma/\lambda_k, \quad k = 1, \ldots, n. \qquad [2.3]$$

where $\sigma$, the surface tension is a constant. This approximation has been seriously questioned in some circumstances (Stuhmiller 1977, Banerjee 1980), but it seems adequate for the present purposes nonetheless.

The drag force on the dispersed phase $k$ is assumed to be of the form

$$\mathbf{M}_k = \kappa\mu_c D(\alpha)\frac{\alpha_k}{\lambda_k^2}(\mathbf{v}_c - \mathbf{v}_k), \qquad [2.4]$$

and the conservation of mixture momentum requires that

$$\mathbf{M}_C + \sum_{k=1}^{n} \mathbf{M}_k = 0. \qquad [2.5]$$

Here

$$\alpha = \alpha_D = \sum_{k=1}^{n} \alpha_k = 1 - \alpha_c, \qquad [2.6]$$

is the total volume fraction of the dispersed phase, $\mu_c$ is the viscosity of the continuous phase, and $\kappa D(\alpha)$ is an empirical factor that describes the viscosity of the mixture. (In particular, $D(0) \equiv 1$, and $\kappa = 9/2$ for a fluid/particle suspension.)

Although much more elaborate assumptions could describe the effects of apparent mass, acceleration etc., [2.4] for the drag is the primary means in this paper of accounting for collisions and interactions in a non-dilute suspension. The magnitude of the drag exerted on the dispersed phase $k$, is assumed to depend only on the instantaneous gross properties of the surrounding suspension measured by the total particulate volume fraction $\alpha$, and fluid velocity of the continuous media, $\mathbf{v}_c$, and, of course, on the size, number and velocity of the droplets of radius $\lambda_k$. This approximation reduces to accepted formulas for a two-phase flow as well as the appropriate single particle limit of a very dilute dispersion. The experimental data of Smith (1965, 1966) confirms that [2.4] is an accurate approximation at least in most circumstances of slow settling.

The entire dispersed phase is characterized by averaged variables that are calculated from the individual species values as follows:

$$\psi_D = \langle \psi_k \rangle = \frac{1}{\alpha} \sum_{k=1}^{n} \alpha_k \psi_k. \qquad [2.7]$$

For example,

$$\mathbf{v}_D = \langle \mathbf{v}_k \rangle, \quad p_D = \langle p_k \rangle; \qquad [2.8]$$

from [2.3], it follows that

$$p_D = p_C + 2\sigma \left\langle \frac{1}{\lambda_k} \right\rangle. \qquad [2.9]$$

The total drag force on the dispersed phase is then

$$\sum_{k=1}^{n} \mathbf{M}_k = \kappa \mu_c \alpha D(\alpha) \left( \frac{\mathbf{v}_c}{a^2} - \left\langle \frac{\mathbf{v}_k}{\lambda_k^2} \right\rangle \right) \qquad [2.10]$$

where

$$\frac{1}{a^2} = \left\langle \frac{1}{\lambda_k^2} \right\rangle. \qquad [2.11]$$

An equal but opposite force must be exerted on the continuous phase, so that

$$\mathbf{M}_c = -\alpha \kappa \mu_c D(\alpha) \left( \frac{\mathbf{v}_c}{a^2} - \left\langle \frac{\mathbf{v}_k}{\lambda_k^2} \right\rangle \right). \qquad [2.12]$$

Since $\sigma$ and $\rho_k (= \rho_D)$ are constants, [2.1] and [2.2] can be written as

$$\frac{\partial}{\partial t} \alpha_k + \nabla \cdot \alpha_k \mathbf{v}_k = 0 \qquad [2.13]$$

$$\rho_D\alpha_k\left(\frac{\partial}{\partial t}\mathbf{v}_k + \mathbf{v}_k\cdot\nabla\mathbf{v}_k\right) = -\alpha_k\nabla p_c + \nabla\cdot\alpha_k\underline{\pi}_k + \rho_D\alpha_k\mathbf{g} + \kappa\mu_c D(\alpha)\frac{\alpha_k}{\lambda_k^2}(\mathbf{v}_c - \mathbf{v}_k), \qquad [2.14]$$

for $k = 1, \ldots, n$.

Since $\alpha_c = 1-\alpha$, mass and momentum conservation in the continuous fluid phase are expressed by

$$-\frac{\partial}{\partial t}\alpha + \nabla\cdot(1-\alpha)\mathbf{v}_c = 0, \qquad [2.15]$$

$$\rho_c(1-\alpha)\left(\frac{\partial}{\partial t}\mathbf{v}_c + \mathbf{v}_c\cdot\nabla\mathbf{v}_c\right) = -(1-\alpha)\nabla p_c + \nabla\cdot(1-\alpha)\underline{\pi}_c + \rho_c(1-\alpha)\mathbf{g} - \kappa\mu_c\alpha D(\alpha)\left(\frac{\mathbf{v}_c}{a^2} - \left\langle\frac{v_k}{\lambda_k^2}\right\rangle\right). \qquad [2.16]$$

Finally, the equations for the total dispersed phase are obtained by summation over index $k$

$$\frac{\partial}{\partial t}\alpha + \nabla\cdot\alpha\mathbf{v}_D = 0, \qquad [2.17]$$

$$\rho_D\alpha\left(\frac{\partial}{\partial t}\mathbf{v}_D + \mathbf{v}_D\cdot\nabla\mathbf{v}_D\right) = -\alpha\nabla p_c + \nabla\cdot\alpha\underline{\pi}_D{}^E + \rho_D\alpha\mathbf{g} + \kappa\alpha\mu_c D(\alpha)\left(\frac{\mathbf{v}_c}{a^2} - \left\langle\frac{\mathbf{v}_k}{\lambda_k^2}\right\rangle\right). \qquad [2.18]$$

where

$$\underline{\pi}_D{}^E = \underline{\pi}_D + \mathbf{v}_D\mathbf{v}_D - \langle\mathbf{v}_k\mathbf{v}_k\rangle \qquad [2.19]$$

is the effective stress.

The extent to which the equations for the entire dispersed phase are dependent on particle sizes is largely measured by the difference

$$\left|\frac{\mathbf{v}_D}{a^2} - \left\langle\frac{\mathbf{v}_k}{\lambda_k^2}\right\rangle\right|.$$

In a two-phase mixture, $k = 1$, $\alpha_1 = \alpha_D$, $\mathbf{v}_D = \mathbf{v}_1$, $a^2 = \lambda_1^2$, $\underline{\pi}_D{}^E = \underline{\pi}_D$; preceding system then reduces to the form given by Ishii (1975).

Constitutive laws for the stress tensors must still be supplied and each of these could be of the usual form

$$\underline{\pi} = \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^+) + \delta(\nabla\cdot\mathbf{v})\underline{I} \qquad [2.20]$$

where the coefficients are empirical functions of mixture properties.

Particles are to be viewed in this formulation as the conceptual limit of droplets when the bulk viscosity of the dispersed liquid becomes infinite and the surface tension approaches zero (as does the effective stress tensor of the discontinuous phase given in the preceding equation).

## 3. FORMULATION FOR A CONTINUOUS SIZE DISTRIBUTION

Let $\phi(\mathbf{r}, t;\lambda)$ be the volume distribution of spherical droplets of radius $\lambda$ at point $\mathbf{r}$ and time $t$. In other words, $\phi(\mathbf{r}, t;\lambda)\,\mathrm{d}\mathbf{r}\,\mathrm{d}\lambda$ is the fraction of the volume $\mathrm{d}\mathbf{r}$, centered at $\mathbf{r}$ at time $t$, which is filled by droplets with radii in the interval $\lambda$ and $\lambda + \mathrm{d}\lambda$. If $n(\mathbf{r}, t;\lambda)$ is the number density of droplets, i.e. the number per unit volume, per unit size, then

$$\phi(\mathbf{r}, t;\lambda) = \frac{4\pi}{3}\lambda^3 n(\mathbf{r}, t;\lambda).$$

(These definitions are appropriate generalizations of those introduced by Bergner 1957.)

The volume fraction of the dispersed phase is then

$$\alpha(\mathbf{r}, t) = \int_{\lambda_s}^{\lambda_l} \phi(\mathbf{r}, t;\lambda)\, \mathrm{d}\lambda \qquad [3.1]$$

where $\lambda_l$ and $\lambda_s$ are the largest and smallest droplet radii in the suspension.

If the entire size range $(\lambda_s, \lambda_l)$ were approximated by $n$ equal intervals of length $\Delta\lambda$ each having particles of the same diameter then the volume fraction of particles of radius $\lambda_k$ is

$$\Delta\alpha_k \simeq \phi(\mathbf{r}, t;\lambda_k)\Delta\lambda\,. \qquad [3.2]$$

Equations [2.13] and [2.14] would then apply to each incremental size interval by replacing $\alpha_k$ by $\Delta\alpha_k$. In the limit, as $\Delta\lambda \to 0$, $\underline{\pi}_k$, $\mathbf{v}_k$, $p_k$ tend to $\underline{\pi}(\mathbf{r}, t; \lambda)$, $\mathbf{v}(\mathbf{r}, t; \lambda)$, $p(\mathbf{r}, t; \lambda)$ and the governing equations become

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\phi\mathbf{v} = 0\,, \qquad [3.3]$$

$$\rho_D\phi\left(\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\phi\nabla p_c + \nabla\cdot\phi\underline{\pi} + \rho_D\phi\mathbf{g} + \kappa\mu_c D(\alpha)\frac{\phi}{\lambda^2}(\mathbf{v}_c - \mathbf{v})\,, \qquad [3.4]$$

with

$$p = p_c + 2\sigma/\lambda\,. \qquad [3.5]$$

Since integration with respect to $\lambda$ replaces summation, the equations for the continuous phase are

$$-\frac{\partial\alpha}{\partial t} + \nabla\cdot(1-\alpha)\mathbf{v}_c = 0 \qquad [3.6]$$

$$\rho_c(1-\alpha)\left(\frac{\partial}{\partial t}\mathbf{v}_c + \mathbf{v}_c\cdot\nabla\mathbf{v}_c\right) = -(1-\alpha)\nabla p_c + \nabla\cdot(1-\alpha)\underline{\pi}_c + \rho_c(1-\alpha)\mathbf{g}$$
$$- \alpha\kappa\mu_c D(\alpha)\left(\frac{\mathbf{v}_c}{a^2} - \frac{1}{\alpha}\int_{\lambda_s}^{\lambda_l}\frac{\phi\mathbf{v}}{\lambda^2}\,\mathrm{d}\lambda\right), \qquad [3.7]$$

where the average particle radius $a$ is defined by

$$\frac{1}{a^2} = \frac{1}{\alpha}\int_{\lambda_s}^{\lambda_l}\frac{\phi\,\mathrm{d}\lambda}{\lambda^2}\,. \qquad [3.8]$$

The total dispersed phase is now characterized by integrated averages as well

$$\left.\begin{matrix} v_D(\mathbf{r}, t) \\ p_D(\mathbf{r}, t) \\ \underline{\pi}_D(\mathbf{r}, t) \end{matrix}\right\} = \frac{1}{\alpha(\mathbf{r}, t)}\int_{\lambda_l}^{\lambda_s} \phi(\mathbf{r}, t;\lambda) \begin{Bmatrix} v(\mathbf{r}, t;\lambda) \\ p(\mathbf{r}, t;\lambda) \\ \underline{\pi}(\mathbf{r}, t;\lambda) \end{Bmatrix} \mathrm{d}\lambda\,, \qquad [3.9]$$

and the equations of motion corresponding to [2.17] and [2.18] are

$$\frac{\partial}{\partial t}\alpha + \nabla\cdot\alpha\mathbf{v}_D = 0\,, \qquad [3.10]$$

$$\rho_D\alpha\left(\frac{\partial}{\partial t}\mathbf{v}_D + \mathbf{v}_D\cdot\nabla\mathbf{v}_D\right) = -\alpha\nabla p_c + \nabla\cdot\alpha\underline{\pi}_D{}^E + \alpha\rho_D\mathbf{g} + \alpha\kappa\mu_c D(\alpha)\left(\frac{\mathbf{v}_c}{a^2} - \frac{1}{\alpha}\int_{\lambda_s}^{\lambda_l}\frac{\phi\mathbf{v}}{\lambda^2}\,\mathrm{d}\lambda\right). \qquad [3.11]$$

Alternatively, the theory for a suspension of distinct particles sizes which constitute separate phases, can be recovered from the foregoing by using the distribution function

$$\phi(\mathbf{r}, t;\lambda) = \sum_{k=1}^{n} \alpha_k(\mathbf{r}, t)\delta(\lambda - \lambda_k).$$

It follows for example that

$$\alpha(\mathbf{r}, t) = \int_{\lambda_s}^{\lambda_l} \phi(\mathbf{r}, t;\lambda)\,\mathrm{d}\lambda = \sum_{k=1}^{n} \alpha_k(\mathbf{r}, t).$$

Since

$$a_k(\mathbf{r}, t) = \int_{\lambda_k-}^{\lambda_k+} \phi(\mathbf{r}, t;\lambda)\,\mathrm{d}\lambda$$

and

$$\alpha_k(\mathbf{r}, t)\mathbf{v}_k(\mathbf{r}, t) = \int_{\lambda_k-}^{\lambda_k+} \mathbf{v}(\mathbf{r}, t;\lambda)\phi(\mathbf{r}, t:\lambda)\,\mathrm{d}\lambda$$

integration of [3.3] over $\lambda_{k^-}$ to $\lambda_{k^+}$ yields [2.13] once again. The other equations follow by similar arguments.

## 4. SEDIMENTATION THEORY

The multi-phase flow theory developed in the preceding sections is illustrated by application to the slow, batch settling of a suspension, Kynch (1952). An initial, uniform suspension of particles (or droplets) of different sizes settles in a tube of finite length. The transient motion of both phases as well as the distribution of particles in the fluid and in the accumulating sediment are to be determined. For droplets, the "sediment" corresponds to a coalesced bulk phase with $\alpha = 1$.

Smith (1965, 1966) and Lockett & Al-Habbooby (1973, 1974) considered the settling of a discrete distribution of particle sizes by employing equations equivalent to those of diffusion theory in which the drift velocity is empirically defined. The latter authors obtained very good agreement between their theory and experiment for a mixture with particles of two different sizes. The use of the two-phase flow equations in these simple cases yields essentially similar results although certain former "assumptions" are now "consequences" in a more fundamental approach. The theory is also in closer agreement with the data of Smith (1966) on a mixture of four particle sizes than his analysis which was based on a fluid envelope model.

Since the motion is slow, which means that the characteristic settling velocity

$$W = \frac{g(\rho_D - \rho_c)\lambda_l^2}{\kappa\mu_c} \qquad [4.1]$$

is small, inertial terms in the momentum equation may be neglected. Viscous stresses and sidewall effects are also assumed negligible in which case the problem reduces to a one-dimensional fluid motion with

$$\mathbf{v} = w\hat{k},\ \mathbf{v}_c = w_c\hat{k} \text{ and } \mathbf{g} = -g\hat{k}.$$

(The general role and effects of the stress tensor will be discussed elsewhere.)

Let $H$ be the height of the container, $\Delta\rho = \rho_D - \rho_c > 0$, and define dimensionless variables by the following transformations: $z \to Hz$; $w \to Ww$; $t \to (H/W)t$; $p_c \to \rho_c g H p_c$; $\phi \to \phi/\lambda_l$; $\lambda \to \lambda_l \lambda$ with $\epsilon = \Delta\rho/\rho_c$. The reduced equations of motion which correspond to [3.3], [3.4] are then

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(\phi w) = 0, \tag{4.2}$$

$$\phi\left[\frac{\partial p_c}{\partial z} + (1+\epsilon) - \epsilon D(\alpha)\left(\frac{w_c - w}{\lambda^2}\right)\right] = 0, \tag{4.3}$$

and

$$\alpha = \int_{\lambda_s}^{1} \phi(r, t; \lambda)\, \mathrm{d}\lambda. \tag{4.4}$$

The equations for the continuous phase, [3.6] and [3.7], are

$$-\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z}(1-\alpha)w_c = 0 \tag{4.5}$$

$$\frac{\partial p_c}{\partial z} + 1 + \frac{\alpha \epsilon D(\alpha)}{1-\alpha}\left(\frac{w_c}{a^2} - \frac{1}{\alpha}\int_{\lambda_s}^{1} \frac{w\phi}{\lambda^2}\, \mathrm{d}\lambda\right) = 0 \tag{4.6}$$

with

$$\frac{1}{a^2} = \frac{1}{\alpha}\int_{\lambda_s}^{1} \frac{\phi}{\lambda^2}\, \mathrm{d}\lambda. \tag{4.7}$$

The dimensionless versions of [3.10] and [3.11] for the entire dispersed phase are:

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z}\alpha w_D = 0 \tag{4.8}$$

$$\frac{\partial p_c}{\partial z} + (1+\epsilon) - \epsilon D(\alpha)\left(\frac{w_c}{a^2} - \frac{1}{\alpha}\int_{\lambda_s}^{1} \frac{\phi w}{\lambda^2}\, \mathrm{d}\lambda\right) = 0. \tag{4.9}$$

The initial conditions are

$$w = 0 = w_c, \quad \phi(z, 0; \lambda) = \Phi(\lambda), \quad \alpha(z, 0) = \bar{\alpha}$$

where $\Phi(\lambda)$ is a prescribed function which is zero for $\lambda < \lambda_s$ or $\lambda > 1$ and $\bar{\alpha}$ is the volume fraction of the uniform suspension. It should be noted that unlike other analyses, our theory applies to any initial, spatially dependent, distribution of particle sizes. For simplicity, however, only an initially uniform suspension is considered here.

The boundary conditions on the top and bottom surfaces of the container are:

$$w = 0 = w_c \text{ on } z = 1 \text{ and } z = 0.$$

The analogous equations for a dispersed phase that consists of spherical particles of $n$ different sizes are obtained, when required, by using the distribution functions

$$\phi(z, t; \lambda) = \sum_{k=1}^{n} \alpha_k(z, t)\delta(\lambda - \lambda_k) \tag{4.10}$$

as described in the last section. (Note again that the radii are ordered in the sequence, $0 < \lambda_s = \lambda_n < \lambda_{n-1} \cdots < \lambda_1 = 1$.) For example, integration of [4.3] over the range $\lambda_{k^-}$ to $\lambda_{k^+}$ yields

$$\alpha_k\left(\frac{\partial p_c}{\partial z}+(1+\epsilon)-\epsilon D(\alpha)\left(\frac{w_c-w_k}{\lambda_k^2}\right)\right)=0$$

where by definition $w(z, t;\lambda_k) = w_k(z, t)$.

The solution of this problem involves four different regions of flow, figure 1, only one of which presents any difficulty. As the suspension settles, there will be a clarified zone, labelled 4 in the figure, bounded by the top plate $z = 1$ and the instantaneous position of the smallest and slowest particle that was originally in contact with this surface. A growing layer of compacted sediment on the bottom plate occupies region 2. The sediment has a maximum volume fraction $\alpha_M < 1$ but for most illustrative purposes $\alpha_M = 1$ is satisfactory. (The value $\alpha_M = 1$ might be especially appropriate for droplets which coalesce upon compaction; $\alpha_M \cong 0.6$ is typical of particle sediments.)

A kinematic shock separates the sediment from region 1 where settling occurs as if there were no end plates and all dependent variables are just constants. Between regions 1 and 4, there is a transitional sector 3 where the volume fractions are variable. Separation is completed when regions 2 and 4 first touch.

Certain general relationships can be obtained by manipulating the equations of motion. Integration of the expression obtained by adding of [4.5] and [4.8] yields

$$(1-\alpha)w_c+\alpha w_D=0 \tag{4.11}$$

which simply states that the net volume flux is always zero. From the definition

$$w_D=\frac{1}{\alpha}\int_{\lambda_s}^{1}\phi w\,d\lambda,$$

it follows that

$$w_c=-\frac{1}{1-\alpha}\int_{\lambda_s}^{1}\phi w\,d\lambda. \tag{4.12}$$

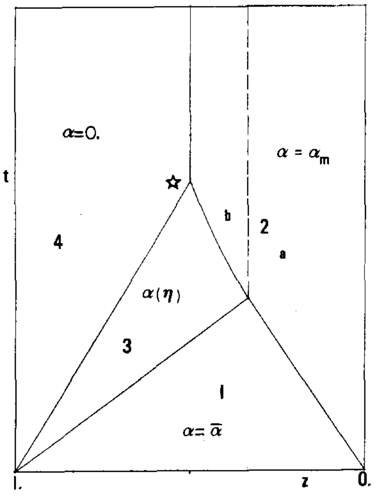


Figure 1. Different regions in the settling of a suspension. 1-uniform settling for a constant volume fraction; 2-the sediment layer, of uniform composition in $a$, of varying composition in $b$, both having a maximum volume fraction $\alpha_M < 1$; 3-transition fan where the volume fraction varies; 4-clarified fluid of the continuous phase with $\alpha = 0$. Separation is completed at point *.

The result of subtracting [4.6] from [4.9] is

$$w_c - \frac{a^2}{\alpha}\int_{\lambda_s}^{1} \frac{\phi w}{\lambda^2}\,d\lambda = \frac{(1-\alpha)a^2}{D(\alpha)}; \qquad [4.13]$$

the substitution of [4.13] into [4.6] yields

$$\frac{\partial p_c}{\partial z} = -1 - \epsilon\alpha, \qquad [4.14]$$

for the pressure. This formula and [4.3] imply that

$$w_c - w = (1-\alpha)\lambda^2/D(\alpha). \qquad [4.15]$$

Replacement of $w$ in [4.12] leads to the important relationship

$$w_c(z,t) = \frac{1-\alpha}{D(\alpha)}\int_{\lambda_s}^{1} \lambda^2\phi(z,t;\lambda)\,d\lambda \qquad [4.16]$$

and it follows that

$$w_D(z,t) = -\frac{(1-\alpha)^2}{\alpha D(\alpha)}\int_{\lambda_s}^{1} \lambda^2\phi(z,t;\lambda)\,d\lambda, \qquad [4.17]$$

$$w(z,t;\lambda) = -\frac{1-\alpha}{D(\alpha)}\left(\lambda^2 - \int_{\lambda_s}^{1} \lambda^2\phi(z,t;\lambda)\,d\lambda\right). \qquad [4.18]$$

All velocities are now given explicitly in terms of the particle distribution function, $\phi$, which must be obtained by solving [4.2] with $w$ given above. This is a peculiar partial differential equation because it involves integrations with respect to a parameter.

Equation [4.18] shows that not every particle settles or moves downward all of the time. As the larger particles sink, the surrounding fluid moves upward to maintain zero net volume transport. This upward flow may be sufficient to drag the smaller particles with it, until they reach a position or conditions where the gravitational pull once again dominates. In particular, particles with radii

$$\lambda^2 < \alpha\bar{\lambda}^2 = \int_{\lambda_s}^{1} \lambda^2\phi\,d\lambda \qquad [4.19]$$

are dragged upward before finally falling to the sediment.

## 5. SETTLING OF A SUSPENSION

We now determine the motion and distribution of particles in each of the flow regions shown in figure 1.

The main region of settling, area 1 of the figure, corresponds to that in an infinitely long container with zero volume transport. The other regions merely adapt this flow to meet the end wall conditions.

The motion in zone 1 is determined from [4.16]–[4.18] by observing that there

$$\phi(z,t;\lambda) = \Phi(\lambda).$$

The particle distribution function in this region is the initial distribution which corresponds to a uniform mixture of constant volume fraction $\bar{\alpha}$. Therefore, the velocities are also constants:

$$\bar{w}_c = \frac{1-\bar{\alpha}}{D(\bar{\alpha})}\int_{\lambda_s}^{1} \lambda^2 \Phi(\lambda)\, d\lambda\,, \tag{5.1}$$

$$\bar{w}_D = -\frac{1-\bar{\alpha}}{\bar{\alpha}}\bar{w}_c\,, \tag{5.2}$$

$$\bar{w} = -\frac{\lambda^2(1-\bar{\alpha})}{D(\bar{\alpha})} + \bar{w}_c\,. \tag{5.3}$$

In the corresponding discrete case, $\bar{\alpha}_k$ is known and

$$\Phi(\lambda) = \sum_{k=1}^{n} \bar{\alpha}_k \delta(\lambda - \lambda_k), \quad \bar{\alpha} = \sum_{k=1}^{n} \bar{\alpha}_k\,. \tag{5.4}$$

Therefore

$$\bar{w}_c = \frac{(1-\bar{\alpha})}{D(\bar{\alpha})}\sum_{k=1}^{n} \lambda_k^{\,2} \bar{\alpha}_k\,, \tag{5.5}$$

and

$$\bar{w}_D = -\frac{(1-\bar{\alpha})}{\bar{\alpha}}\bar{w}_c\,, \tag{5.6}$$

$$\bar{w}_k = -\frac{(1-\bar{\alpha})}{D(\bar{\alpha})}\lambda_k^{\,2} + \bar{w}_c\,. \tag{5.7}$$

The falling particles are brought to rest across a kinematic shock that bounds the sedimentary layer, region 2, whose thickness increases with time. The shock condition is derived from [4.2].

If $U$ is the velocity of a kinematic shock and a particle crosses the discontinuity from the front (or plus side) to the back (or minus side) then the conservation of mass requires that

$$[(w - U)\phi]_-^+ = (w^+ - U)\phi^+ - (w^- - U)\phi^- = 0\,. \tag{5.8}$$

Only one such shock is required to bring *all* particles to rest in the sediment and the velocity $U$ is independent of $\lambda$. Since the sediment is characterized by the values

$$\alpha^- = \alpha_M, \;\; w^- = 0\,, \tag{5.9}$$

where

$$\alpha^{\pm} = \int_{\lambda_s}^{1} \phi^{\pm}\, d\lambda\,, \tag{5.10}$$

[5.8] can be rearranged and integrated to yield

$$U = -\,w_D^{\;+}\alpha^+/(\alpha_M - \alpha^+)\,. \tag{5.11}$$

The shock speed is exactly that for conventional two-phase flow theory in which the dispersed phase is defined by an average particle size.

The distribution of particles in the sediment is determined from [5.8] by solving for $\phi^-$ since all other quantities are known. For the discrete suspension, this calculation yields

$$\alpha_k^{\;-} = \alpha_k^{\;+}(U - w_k^{\;+})/U\,. \tag{5.12}$$

(For an emulsion, the "sedimentary" layer is taken to be a condensation of the dispersed phase with $\alpha \sim 1$ as would be the case if droplet coalescence occurs.) Equations [5.11] and [5.12] hold in general across the sediment boundary whether the conditions in front of the shock are constant or variable.

The clarified continuous fluid phase, region 4 of figure 1, is bounded by the top plate and the locus of the smallest particle that was originally in contact with this surface. The trajectory of this particle is the solution of

$$\frac{dz}{dt} = w(z, t; \lambda_s)$$

with $z = 1$ at $t = 0$, and $w$ is given by [4.18]. However, the explicit formula for the particle velocity requires the determination of the distribution function in the last remaining zone 3, the interval between the largest and the smallest particles that were both at the end plate $z = 1$, at time zero.

There are some significant differences between the discrete and continuous particle distributions in sector 3 (although not necessarily in the limit at $n \to \infty$). For example, a distribution of $n$ distinct particle sizes implies $n$ kinematic shocks which are essentially trajectories of the "last" particles of each type. Discontinuities can occur with a continuous size distribution as well but none develop in this particular region (in this problem).

The discrete case is in fact the simpler of the two to treat since between each pair of shocks the flow is uniform and constant. For this reason, it is discussed first but with minimal detail because the analysis is quite similar to that of Smith (1966) whose equations were based on a spherical fluid envelope model.

The transition region 3 consists of a series of kinematic shocks emanating from the point $z = 1$, $t = 0$ and separated by sectors of constant conditions. The shock moving with constant velocity $U_k$ is the trajectory of the particle of radius $\lambda_k$ initially at $z = 1$. Behind this front, which is also a particle path, $\alpha_k = \alpha_k^- = 0$.

The transition zone resembles a fan made up of distinct sectors which are labelled sequentially. Let the region

$$1 + U_m t < z < 1 + U_{m+1} t$$

(where $U_m$ are all negative) be designated sector $m + 1$; sector 1 is then region 1 of figure 1. Within the fan, the distribution function is

$$\phi = \sum_{k=1}^{n} \alpha_k(m)\delta(\lambda - \lambda_k), \quad m = 1, \ldots, n \tag{5.13}$$

where

$$\alpha_k(m) \equiv 0 \text{ for } k < m. \tag{5.14}$$

Here $m = n + 1$ is actually region 4 of clarified fluid. The volume fractions and velocities in sector $m$ are obtained from [5.5] to [5.7]:

$$\alpha(m) = \sum_{j=1}^{n} \alpha_j(m) \tag{5.15}$$

$$w_c(m) = \frac{1 - \alpha(m)}{D(\alpha(m))} \sum_{k=1}^{n} \lambda_k^2 \alpha_k(m) \tag{5.16}$$

$$w_D(m) = -\frac{1-\alpha(m)}{\alpha(m)} w_c(m) \qquad [5.17]$$

$$w_k(m) = -\frac{1-\alpha(m)}{D(\alpha(m))}\lambda_k^2 + w_c(m)\,. \qquad [5.18]$$

Since $w_k(m)$ is of no importance when $\alpha_k(m) \equiv 0$, for notational convenience, it too is assigned a zero value for $k < m$.

With $U_m = U(m)$ the kinematic shock condition [5.8] across the front, $z = 1 + U(m)t$ may be written as

$$\alpha_k(m+1)(w_k(m+1) - U(m)) = \alpha_k(m)(w_k(m) - U(m))\,, \qquad [5.19]$$

where $\alpha_k(m+1) = \alpha_k^-$, $\alpha_k(m) = \alpha_k^+$, etc. For $k = m$, the preceding formula implies that

$$U(m) = w_m(m)\,, \qquad [5.20]$$

since $\alpha_m(m+1) \equiv 0$, by [5.14]. Thus the shock velocity is the particle velocity of the "last" particle of that size.

To find the solution we start with the known values in region 1 obtained from [5.5], [5.7] and [5.11]:

$$\alpha_k(1) = \bar{\alpha}_k,\; w_k(1) = \bar{w}_k,\; U_{(1)} = w_1(1), \text{ etc.} \qquad [5.21]$$

The shock condition [5.19] and [5.18] provide a nonlinear set of algebraic equations for the values of $\alpha_k(2)$, which determine $w_k(2)$, $w_c(2)$, $w_D(2)$, etc. The procedure is then repeated until region 4 of clarified fluid is reached after crossing the last shock, that is, the locus of the smallest particle of the system.

The system of equations is solved at every stage by iteration; there are no special difficulties. The results for a suspension of 4 particle sizes with $\bar{\alpha}_k = \bar{\alpha}/4$, $\bar{\alpha} = 0.2$, and $\lambda_s = 0.4$, are given in table 1, for the drag law

$$D(\alpha) = \left(1 - \frac{\alpha}{\alpha_M}\right)^{-2} \text{ and } \alpha_M = 0.6 \qquad [5.22]$$

We consider next the solution for a continuous distribution of particle sizes. In this case, the distribution function $\phi$ (see [4.2] *et seq.*) must be determined in the transition zone by solving

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\phi w = 0 \qquad [5.23]$$

Table 1. Volume fractions $\alpha_k(m)$, and velocities $w_k(m)$ for a suspension with four particle sizes (1, 0.8, 0.6, 0.4) and the drag law $D(\alpha) = (1 - 5/3\alpha)^{-2}$

| $m$ \ $k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | −0.3172<br>0.0500 | −0.1892<br>0.0500 | −0.0896<br>0.0500 | −0.0185<br>0.0500 |
| 2 | 0 | −0.2272<br>0.0711 | −0.1147<br>0.0562 | −0.0344<br>0.0528 |
| 3 | 0 | 0 | −0.1548<br>0.0873 | −0.0577<br>0.0601 |
| 4 | 0 | 0 | 0 | −0.0931<br>0.0945 |

Key: upper entry $w_k(m)$; lower entry $\alpha_k(m)$

with

$$w(z, t;\lambda) = -\frac{1-\alpha}{D(\alpha)}\left(\lambda^2 - \int_{\lambda_s}^{1} \lambda^2 \phi(z, t;\lambda)\, \mathrm{d}\lambda\right), \qquad [5.24]$$

subject to the condition

$$\phi = \Phi(\lambda) \qquad [5.25]$$

on

$$1 - z = \int_0^t w(z, t;1)\, \mathrm{d}t = -\frac{1-\bar{\alpha}}{D(\bar{\alpha})}\left(1 - \int_{\lambda_s}^{1} \lambda^2 \Phi(\lambda)\, \mathrm{d}\lambda\right)t. \qquad [5.26]$$

The last equation describes the trajectory of the largest particle which was initially at the top plate $z = 1$. Were it not for the integration with respect to $\lambda$ the solution would be a straightforward application of the method of characteristics. As it stands, the problem is a rather peculiar combination of derivatives and integrals. However, the solution can be found by introducing a similarity variable. Let

$$\eta = \frac{1-z}{t} \qquad [5.27]$$

and assume that

$$\phi(z, t;\lambda) = \psi(\eta;\lambda) \qquad [5.28]$$

so that

$$-w(z, t;\lambda) = \omega(\eta;\lambda) = \frac{1-\alpha}{D(\alpha)}\left(\lambda^2 - \int_{\lambda_s}^{1} \lambda^2 \psi(\eta;\lambda)\, \mathrm{d}\lambda\right), \qquad [5.29]$$

$$\alpha = \int_{\lambda_s}^{1} \psi(\eta;\lambda)\, \mathrm{d}\lambda. \qquad [5.30]$$

Equation [5.23] may then be written as

$$(\omega - \eta)\frac{\mathrm{d}\psi}{\mathrm{d}\eta} + \frac{\mathrm{d}\omega}{\mathrm{d}\eta}\psi = 0 \qquad [5.31]$$

with

$$\psi = \Phi(\lambda) \qquad [5.32]$$

on the ray

$$\bar{\eta} = \bar{\omega}(\bar{\eta};1) = \frac{1-\bar{\alpha}}{D(\bar{\alpha})}\left(1 - \int_{\lambda_s}^{1} \lambda^2 \Phi\, \mathrm{d}\lambda\right). \qquad [5.33]$$

For each $\lambda$, the problem must be solved in the sector

$$\omega(\bar{\eta};\lambda) = \bar{\eta}(\lambda) \leq \eta \leq \bar{\eta} = \omega(\bar{\eta};1). \qquad [5.34]$$

The differential equation has a singular point at $\bar{\eta}(\lambda)$ which represents the locus of the particle of radius $\lambda$ falling from the top plate.

In order to calculate the distribution function, [5.31] is first converted to an integral equation

$$\psi(\eta;\lambda)=\Phi(\lambda)\exp\int_{\bar{\eta}}^{\eta}\frac{d\omega(\zeta;\lambda)}{d\zeta}\frac{d\zeta}{\zeta-\omega(\zeta;\lambda)} \qquad [5.35]$$

where $\omega(\eta;\lambda)$ is given above. The solution may be obtained by iteration and a successful algorithm is given in appendix A.

The results of one iteration are the approximate formulas

$$\psi(\eta;\lambda)\simeq\Phi(\lambda)H(\sqrt{(\eta)}-\lambda), \qquad [5.36]$$

and

$$\alpha(\eta)\simeq\int_{\lambda_s}^{\sqrt{\eta}}\Phi(\lambda)\,d\lambda, \qquad [5.37]$$

$$\omega(\eta;\lambda)\simeq\frac{1-\alpha}{D(\alpha)}\left(\lambda^2-\int_{\lambda_s}^{\sqrt{\eta}}\lambda^2\Phi(\lambda)\,d\lambda\right), \qquad [5.38]$$

$$w_c\simeq\frac{1-\alpha}{D(\alpha)}\int_{\lambda_s}^{\sqrt{\eta}}\lambda^2\Phi(\lambda)\,d\lambda, \qquad [5.39]$$

in the sector defined by [5.34]. $H(x)$ is the Heaviside function.

## 6. DISCUSSION

To illustrate theory, the simple settling problem of a spatially homogeneous mixture has been solved for both discrete and continuous distributions of particle and droplet sizes. In the former case, a discretization of the distribution function is equivalent to assuming that diameter of a particle must be one of only a finite number of possibilities. Collectively, the identical particles of different sizes constitute separate phases or species. The individual or partial volume fractions can then be calculated directly in every region of figure 1. In particular transition zone 3 consists of kinematic shocks separated by sectors where constant conditions prevail.

Figure 2 illustrates the settling process for a uniform suspension which consists of four

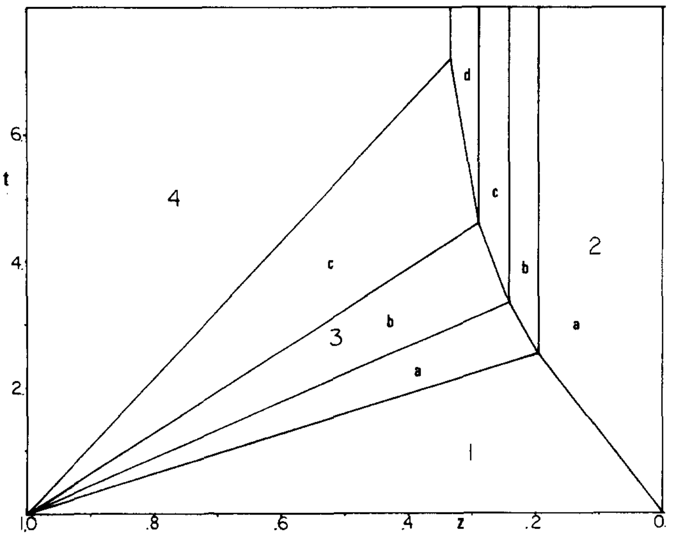


Figure 2. Sedimentation of a suspension of four particle sizes (1, 0.8, 0.6, 0.4) and the drag law $D=(1-[5/3]\alpha)^{-2}$. Values of the volume fractions in each region ($\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$), $\alpha$, are: 1-(0.05, 0.05, 0.05, 0.05), 0.2; 2*a*-(0.257, 0.173, 0.108, 0.062), 0.6; 2*b*-(0.0, 0.349, 0.167, 0.084), 0.06; 2*c*-(0.0, 0.0, 0.448, 0.152), 0.6; 2*d*-(0.0, 0.0, 0.0, 0.6), 0.6; 3*a*-(0.0, 0.071, 0.056, 0.053), 0.180; 3*b*-(0.0, 0.0, 0.087, 0.060), 0.147; 3*c*-(0.0, 0.0, 0.0, 0.095), 0.095; 4-$\alpha=0$.

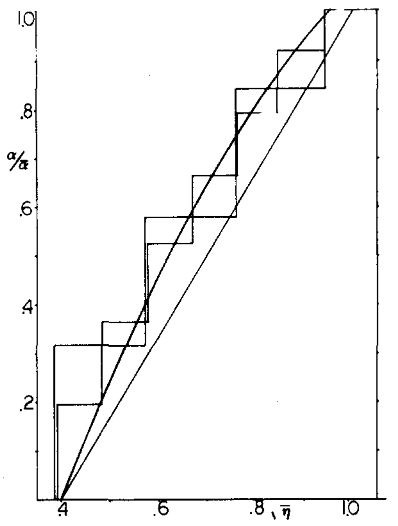


Figure 3. A comparison of discrete and continuous approximations for the volume fraction $\alpha$ in the transition fan based on 4 and 7 particle size species and the first two iterations. Initially $\bar{\alpha} = 0.2$; $\lambda_5 = 0.4$ and $D(\alpha) = 1 - \alpha$.

particle sizes $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 0.8, 0.6, 0.4)$ each diameter represented by an initial volume fraction of 0.05 (so that $\bar{\alpha} = 0.2$). The drag law is $D(\alpha) = (1 - (5/3)\alpha)^{-2}$ which is typical of droplets and particles, Ishii & Chawla (1979). The velocities and volume fractions in this case are recorded in table 1. The calculation is readily made for any drag law (or any initial, spatially dependent distribution).

The velocity and volume fraction of every size species generally increases across each of the shocks of the transition fan, excepting, of course, that discontinuity which is the final trajectory of particles of that diameter. However, the total volume fraction $\alpha$ of the dispersed phase decreases by steps to the value zero in the clarified fluid. Increasing the drag gives a slower rate of descent as might be expected.

The results of the two phase flow theory on mixtures with two particle species essentially reproduce those of Lockett & Al-Habbooby (1973) (who used an equivalent of a diffusion theory) and are in very good agreement with the data of Smith (1965). The theoretical results for a mixture of four size species also agree closely with the experimental data cited by Smith (1966); comparison with experiments on continuous distributions is yet to be made.

The approximate solution for a continuous distribution of particle sizes is based on the iteration formulas in the appendix. For an initial distribution which is a constant and the analog of one of the discrete cases cited above, equations [A11] are obtained at the first iteration for variables in the transition sector. In particular,

$$\alpha(\eta) \simeq \bar{\Phi}(\sqrt{(\eta)} - \lambda_s)$$

i.e. at this stage the volume fraction is a linear function of $\sqrt{\eta}$. A second iteration leads to a more complicated formula, [A15].

With $D(\alpha) = 1 - \alpha$, the first two iterations for the volume fraction $\alpha$ and the discrete approximations corresponding to four and seven particle sizes are compared in figure 3. The trend seems to indicate apparent convergence of both methods.

The calculations based on discrete approximations of the distribution function are the easiest to implement and should be very accurate if the number of particle species is appropriately large, say seven or more. However, the model with a continuous distribution of particle sizes may have advantages in the less than ideal circumstances of most two phase flow applications.

## APPENDIX A

The problem for a continuous distribution of particle sizes is embodied in the integral equation [5.35]. The solution is obtained by iteration.

Assume $\psi_{(n)}(\eta;\lambda)$ the $n$th iterant is known. Calculate

$$\alpha_{(n)} = \int_{\lambda_s}^{1} \psi_{(n)}\,d\lambda \qquad [A1]$$

and

$$\omega_{(n)}(\eta;\lambda) = \frac{1-\alpha_{(n)}}{D(\alpha_{(n)})}\left(\lambda^2 - \int_{\lambda_s}^{1} \lambda^2 \psi_{(n)}\,d\lambda\right). \qquad [A2]$$

Determine next by separate iterations in

$$\tilde{\eta}_{(n)} = \omega_{(n)}(\tilde{\eta}_{(n)};\lambda); \quad \bar{\eta}_{(n)} = \omega_{(n)}(\bar{\eta}_{(n)};1), \qquad [A3]$$

the approximate boundaries of the transition fan

$$\tilde{\eta}_{(n)} < \eta < \bar{\eta}_{(n)}. \qquad [A4]$$

At this stage, the velocities of the continuous and discrete phases are

$$-\frac{\alpha_{(n)}}{1-\alpha_{(n)}} w_{D_{(n)}} = w_{C_{(n)}} = \frac{1-\alpha_{(n)}}{D(\alpha_{(n)})}\int_{\lambda_s}^{1} \lambda^2 \phi_{(n)}\,d\lambda. \qquad [A5]$$

The next iterant is

$$\psi_{(n+1)} = \Phi(\lambda)\exp\int_{\tilde{\eta}_{(n)}}^{\eta} \frac{d\omega_{(n)}}{d\xi}\frac{d\xi}{\xi-\omega_{(n)}} \qquad [A6]$$

and the procedure starts anew. The exact solution is supposedly obtained in the limit $\lim_{n\to\infty}\psi_{(n)}$ but as a practical matter only one or two iterations seem possible.

The results of one iteration, with $\psi_{(0)} \equiv 0$, are the approximate formulas

$$\psi(\eta;\lambda) \simeq \Phi(\lambda) H(\surd(\eta) - \lambda) \qquad [A7]$$

and

$$\alpha(\eta) \simeq \int_{\lambda}^{\surd\eta} \Phi(\lambda)\,d\lambda \qquad [A8]$$

$$\omega(\eta;\lambda) \simeq \frac{1-\alpha}{D(\alpha)}\left(\lambda^2 - \int_{\lambda_s}^{\surd\eta} \lambda^2 \Phi(\lambda)\,d\lambda\right)$$

$$w_C \simeq \frac{1-\alpha}{D(\alpha)}\int_{\lambda_s}^{\surd\eta} \lambda^2 \Phi(\lambda)\,d\lambda \qquad [A9]$$

in the sector defined by

$$\omega(\tilde{\eta};\lambda) = \tilde{\eta} < \eta < \bar{\eta} = \omega(\bar{\eta};1).$$

For an initial distribution function which is a constant,

$$\Phi(\lambda) = \bar{\Phi} = \bar{\alpha}/(1-\lambda_s) \qquad [A10]$$

we have

$$\left.\begin{aligned} \alpha &\simeq \bar{\Phi}(\sqrt{(\eta)}-\lambda_s) \\ \omega &\simeq \frac{1-\alpha}{D(\alpha)}\left(\lambda^2-\frac{\bar{\Phi}}{3}(\eta^{3/2}-\lambda_s^3)\right) \\ w_c &\simeq \frac{1-\alpha}{D(\alpha)}\frac{\bar{\Phi}}{3}(\eta^{3/2}-\lambda_s^3). \end{aligned}\right\}, \qquad \text{[A11]}$$

If $D(\alpha)=1-\alpha$ in this particular case, the second iteration is:

$$\psi_{(2)}(\eta;\lambda)=\bar{\Phi}\exp\int_{\bar{\eta}}^{\eta}\frac{d\omega_{(1)}}{d\zeta}\frac{d\zeta}{\zeta-\omega_{(1)}}=\bar{\Phi}\exp J(\eta;\lambda) \qquad \text{[A12]}$$

for

$$\tilde{\eta}_{(1)}<\eta<\bar{\eta}_{(1)}$$

where

$$\omega_{(1)}(\eta;\lambda)=\lambda^2-\frac{\Phi}{3}(\eta^{3/2}-\lambda_s^3)$$

and

$$\tilde{\eta}_{(1)}=\omega_{(1)}(\tilde{\eta},\lambda), \quad \bar{\eta}_{(1)}=\omega_{(1)}(\bar{\eta};1).$$

If for convenience the cumbersome iterative subscript notation is dropped then

$$J(\eta;\lambda)=-\frac{\bar{\Phi}}{2}\int_{\tilde{\eta}}^{\eta}\frac{\zeta^{1/2}\,d\zeta}{\zeta-\lambda^2+\frac{\bar{\Phi}}{3}(\eta^{3/2}-\lambda_s^3)}. \qquad \text{[A13]}$$

This may be integrated in closed form but the formula is complicated and some additional notation is required. If

$$\xi=\eta^{1/2}=(1-z)^{1/2}$$

$$\sigma=\sqrt{[\tilde{\eta}(\lambda)]}$$

and

$$A(\sigma)=\bar{\Phi}\sigma/((2+\bar{\Phi}\sigma))$$

$$B(\sigma)=(3+\bar{\Phi}\sigma)/(2+\bar{\Phi}\sigma)$$

$$q(\sigma)=(3(3+\bar{\Phi}\sigma)(1-\bar{\Phi}\sigma))^{1/2}$$

$$C(\sigma)=-3B(\sigma)/q(\sigma)$$

$$\mathscr{F}(r;\sigma)=\bar{\Phi}r^2+3(1+\bar{\Phi}\sigma)r+3(\bar{\Phi}\sigma^2+2\sigma)$$

$$\mathscr{G}(r;\sigma)=\frac{2\bar{\Phi}r+3(1+\bar{\Phi}\sigma)-q(\sigma)}{2\bar{\Phi}r+3(1+\bar{\Phi}\sigma)+q(\sigma)}$$

then

$$\psi(\xi;\lambda)=\left(\frac{\tilde{\eta}-\sigma}{\xi-\sigma}\right)^{A(\sigma)}\left(\frac{\mathscr{F}(\tilde{\eta}-\sigma;\sigma)}{\mathscr{F}(\xi-\sigma;\sigma)}\right)^{B(\sigma)}\left(\frac{\mathscr{G}(\eta-\sigma;\sigma)}{\mathscr{G}(\xi-\sigma;\sigma)}\right)^{C(\sigma)} \qquad \text{[A14]}$$

and

$$\alpha(\eta) = \bar{\Phi}\int_{\lambda_s}^{\xi} \frac{\left(1+\frac{\bar{\Phi}}{2}\sigma\right)\sigma\psi(\xi;\lambda)}{\left(\sigma^2+\frac{\bar{\Phi}}{3}(\sigma^3-\lambda_s^3)\right)^{1/2}}\,d\sigma \qquad [A15]$$

The last expression which must be computed numerically was used to determine the curve of figure 3.

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